

Applications of Some Elliptic Equations in Riemannian Manifolds *

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Abstract

Let (M^{n+1}, g) be a compact Riemannian manifold with smooth boundary B and non-negative Bakry-Emery Ricci curvature. In this paper, we use the solvability of some elliptic equations to prove some estimates of the weighted mean curvature and some related rigidity theorems. As their applications, we obtain some lower bound estimate of the first nonzero eigenvalue of the drifting Laplacian acting on functions on B and some corresponding rigidity theorems.

Keywords: Riemannian manifold, Bakry-Emery Ricci curvature, Rigidity theorem, Eigenvalue, Weighted mean curvature

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1 Introduction and main results

Let (M^{n+1}, g) be a compact Riemannian manifold with smooth boundary $B^n = \partial M$. Let dV and dA be the canonical measures on M and B respectively, V and A be the volume of M and the area of B . Given $f \in C^\infty(M)$, Reilly's formula [14] states that

$$\int_M ((\bar{\Delta}f)^2 - |\bar{\nabla}^2 f|^2 - Ric(\bar{\nabla}f, \bar{\nabla}f))dV = \int_B (2(\Delta f)f_\nu + nH(f_\nu)^2 + \Pi(\nabla f, \nabla f))dA, \quad (1.1)$$

here $\bar{\nabla}f$, $\bar{\Delta}f$, $\bar{\nabla}^2 f$ being the gradient, the Laplacian and the Hessian of f on M , Ric the Ricci curvature of M , ∇f and Δf the gradient and the Laplacian of f in B , and $\Pi(X, Y) = g(\bar{\nabla}_X \nu, Y)$ for $\forall X, Y \in TB$ and $H = \frac{1}{n}tr\Pi$ the second fundamental form and the mean curvature of B with respect to the outer unit normal ν on B .

Reilly [13] used the formula (1.1) to prove that if M has non-negative Ricci curvature with convex boundary and $H \geq \frac{A}{(n+1)V}$, then M is isometric to an Euclidean ball. Later Ros [12] removed the condition of the convex boundary and obtains the same conclusion.

Recently Ma and Du [8] studied the drifting laplacian operator $\bar{\Delta}_h = \bar{\Delta} - \bar{\nabla}h \cdot \bar{\nabla}$ for a smooth function h on M . This operator is self-adjoint operator with respect to the weighted measure

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$dV_h = e^{-h}dV$. They extended the above Reilly's formula and showed that

$$\int_M ((\bar{\Delta}_h f)^2 - |\bar{\nabla}^2 f|^2 - Ric_h(\bar{\nabla} f, \bar{\nabla} f))dV_h = \int_B (2(\Delta_h f)f_\nu + nH_h(f_\nu)^2 + \Pi(\nabla f, \nabla f))dA_h \quad (1.2)$$

here $Ric_h = Ric + \bar{\nabla}^2 h$, $\Delta_h f = \Delta f - \nabla h \cdot \nabla f$, $H_h = H - \frac{1}{n}h_\nu$ and $dA_h = e^{-h}dA$. In [17], the second author proved the sharp gradient estimate for positive solution of the Laplacian with a general drift B , i.e. $\bar{\Delta} f - Bf = 0$, where B is a vector field.

Using the inequalities of $|\bar{\nabla}^2 f|^2 \geq \frac{1}{n+1}(\bar{\Delta} f)^2$ and $\frac{1}{n+1}a^2 + \frac{1}{m-n-1}b^2 \geq \frac{1}{m}(a-b)^2$, we know that the equation (1.2) become the following inequality:

$$\int_M \frac{m-1}{m}((\bar{\Delta}_h f)^2 - Ric_m(\bar{\nabla} f, \bar{\nabla} f))dV_h \geq \int_B (2(\Delta_h f)f_\nu + nH_h(f_\nu)^2 + \Pi(\nabla f, \nabla f))dA_h \quad (1.3)$$

here $Ric_m = Ric_h - \frac{1}{m-n-1}\bar{\nabla} h \otimes \bar{\nabla} h$, $m \geq n+1$, and $m = n+1$ if and only if h is a constant. This curvature tensor is called Bakry-Emery Ricci curvature (see [5]). With help of this inequality, Ma and Du obtained the lower bound for the first eigenvalue of the drifting Laplacian on the compact manifold with positive Bakry-Emery Ricci curvature (see [8]). It states that if $Ric_m \geq (m-1)K > 0$, $H_h \geq 0$ or $\Pi \geq 0$, then the first Neumann eigenvalue $\lambda_1^N(\bar{\Delta}_h) \geq mK$, or the first Dirichlet eigenvalue $\lambda_1^D(\bar{\Delta}_h) \geq mK$. This conclusion is a generalization of Reilly's [13] and Escobar's results [2]. Recently Li and Wei [6] proved that this result is sharp, i.e. if $\lambda_1^D(\bar{\Delta}_h) = mK$ or $\lambda_1^N(\bar{\Delta}_h) = mK$, then the manifold is isometric to a Euclidean hemisphere. They extended the rigidity theorem of Reilly [13] and Escobar [2]. From Theorem 5 in [11], we know that if $Ric_m \geq (m-1)K > 0$, then M is compact and the diameter $diam(M) \leq \frac{\pi}{\sqrt{K}}$. In [16], the second author proved that if $Ric_m \geq (m-1)K > 0$ and $diam(M) = \frac{\pi}{\sqrt{K}}$, then M is isometric to a Euclidean sphere of radius $\frac{1}{\sqrt{K}}$.

Based on the Reilly formula (1.1), Ros [12] showed an estimate of the mean curvature. Similarly using the Reilly type inequality (1.3) we may extend Ros's result to the manifold with nonnegative Bakry-Emery Ricci curvature and obtain an estimate of the weighted mean curvature. However in this paper we do not use the Reilly type formula but the divergence theorem to prove this result.

Theorem 1.1. *Let (M^{n+1}, g) be a compact Riemannian manifold with smooth boundary B and $Ric_m \geq 0$. If the weighted mean curvature $H_h > 0$, then*

$$\int_B \frac{dA_h}{H_h} \geq \frac{mn}{m-1}V_h, \quad (1.4)$$

here $V_h = \int_M dV_h$ denotes the weighted volume of M . The equality holds if and only if M is isometric to an Euclidean ball and h is constant.

Let $A_h = \int_B dA_h$, if $H_h \geq \frac{(m-1)A_h}{mnV_h}$, then the equality in (1.4) holds. Thus we easily deduce the following rigidity theorem.

Corollary 1.2. *Let (M^{n+1}, g) be a compact Riemannian manifold with smooth boundary B and $Ric_m \geq 0$. If the weighted mean curvature $H_h \geq \frac{(m-1)A_h}{mnV_h}$, then M is isometric to an Euclidean ball and h is constant.*

Remark 1: When h is a constant, the Bakry-Emery Ricci curvature and the weighted mean curvature become the classical Ricci curvature and the mean curvature respectively, and $m = n+1$. In this case, Corollary 1.2 is Ros's result [12].

Using the similar method we prove the following estimate of the weighted mean curvature and the related rigidity theorem. If h is a constant, then it is a Reilly's result in [13].

Theorem 1.3. *Let (M^{n+1}, g) be a compact Riemannian manifold with convex boundary B and $Ric_m \geq 0$. Then*

$$\int_B H_h dA_h \leq \frac{(m-1)A_h^2}{mnV_h}, \quad (1.5)$$

The equality holds if and only if M is isometric to an Euclidean ball and h is constant.

Now we discuss some applications of Corollary 1.2. Firstly we prove that if the manifold has nonnegative Bakry-Emery Ricci curvature, then the critical point of the weighted isoperimetric functional is an Euclidean ball. Recently there are many results about isoperimetric problems on the manifold with density, for example, see [1], [3], [4], [9],[10], [15] and so on.

Theorem 1.4. *Let (M^{n+1}, g) be a compact Riemannian manifold with $Ric_m \geq 0$, and let Ω be a compact domain in M with smooth boundary $\partial\Omega$. If Ω is a critical point of the weighted isoperimetric functional*

$$\Omega \rightarrow \frac{A_h(\partial\Omega)^m}{V_h(\Omega)^{m-1}},$$

then Ω is isometric to an Euclidean ball and h is constant.

In [18], Xia used the Reilly's formula (1.1) and Ros's result[12] to obtain a lower bound of the first nonzero eigenvalue $\lambda_1(\Delta)$ of the Laplacian acting on functions on B and the corresponding rigidity theorem.

Theorem 1.5. (Xia's theorem in [18]) *Let (M^{n+1}, g) be a compact Riemannian manifold with nonempty boundary B and nonnegative Ricci curvature. If the second fundamental form of B satisfies $\Pi \geq cI$ (in the matrix sense), then $\lambda_1(\Delta) \geq nc^2$. The equality holds if and only if M is isometric to an Euclidean ball.*

In this paper, we use different method to generalize this result to the drifting Laplacian $\Delta_h = \Delta - \nabla h \cdot \nabla$. Our proof is mainly based on the divergence theorem and Corollary 1.2.

Theorem 1.6. *Let (M^{n+1}, g) be a compact Riemannian manifold with nonempty boundary B and $Ric_m \geq 0$. If the second fundamental form of B satisfies $\Pi \geq cI$ (in the matrix sense) and $h_\nu \leq -(m-n-1)c$, then $\lambda_1(\Delta_h) \geq (m-1)c^2$. The equality holds if and only if M is isometric to an Euclidean ball of radius $\frac{1}{c}$ and h is constant.*

Using the similar method in the proof of Theorem 1.6, we also show the following result.

Theorem 1.7. *Let (M^{n+1}, g) be a compact Riemannian manifold with nonempty boundary B and $Ric_m \geq 0$. If the second fundamental form of B satisfies $\Pi \geq cI$ (in the matrix sense), and the weighted mean curvature $H_h \geq \frac{\lambda_1(\Delta_h)}{nc}$, then $\lambda_1(\Delta_h) \leq (m-1)c^2$.*

When h is a constant, we obtain the following corollary.

Corollary 1.8. *Let (M^{n+1}, g) be a compact Riemannian manifold with nonempty boundary B and nonnegative Ricci curvature. If the second fundamental form of B satisfies $\Pi \geq cI$ (in the matrix sense) and the mean curvature $H \geq \frac{\lambda_1(\Delta)}{nc}$, then $\lambda_1(\Delta) \leq nc^2$.*

Combining Corollary 1.8 with Theorem 1.5, we know that under the condition of Corollary 1.8 the manifold M is isometric to an Euclidean ball. This is the Theorem 3 in [18] proved by Xia.

Corollary 1.9. (Xia's theorem in [18]) *Let (M^{n+1}, g) be a compact Riemannian manifold with nonempty boundary B and nonnegative Ricci curvature. If the second fundamental form of B satisfies $\Pi \geq cI$ (in the matrix sense) and the mean curvature $H \geq \frac{\lambda_1(\Delta)}{nc}$, then M is isometric to an Euclidean ball of radius $\frac{1}{c}$.*

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2 Proof of Theorem 1.1, 1.3 and 1.4

Proof of Theorem 1.1. Let f be the smooth solution of the Dirichlet problem

$$\begin{cases} \bar{\Delta}_h f &= 1 & \text{in } M, \\ f &= 0 & \text{on } B. \end{cases}$$

Suppose $F = \frac{1}{2}|\bar{\nabla} f|^2 - \frac{1}{m}f$, then we have that

$$\begin{aligned} \bar{\Delta}_h F &= |\bar{\nabla}^2 f|^2 + \bar{\nabla} \bar{\Delta}_h f \bar{\nabla} f + Ric_h(\bar{\nabla} f, \bar{\nabla} f) - \frac{1}{m} \bar{\Delta}_h f \\ &\geq \frac{1}{n+1}(\bar{\Delta}_h f)^2 + \frac{1}{m-n-1}(\bar{\nabla} h \bar{\nabla} f)^2 + \bar{\nabla} \bar{\Delta}_h f \bar{\nabla} f + Ric_m(\bar{\nabla} f, \bar{\nabla} f) - \frac{1}{m} \bar{\Delta}_h f \\ &\geq \frac{1}{m}(\bar{\Delta}_h f)^2 + \bar{\nabla} \bar{\Delta}_h f \bar{\nabla} f + Ric_m(\bar{\nabla} f, \bar{\nabla} f) - \frac{1}{m} \bar{\Delta}_h f \\ &\geq \frac{1}{m}(\bar{\Delta}_h f)^2 + Ric_m(\bar{\nabla} f, \bar{\nabla} f) - \frac{1}{m} \bar{\Delta}_h f \geq 0. \end{aligned} \tag{2.1}$$

Integrating both sides of the above inequality on M with respect to the weighted measure dV_h , we have that

$$\int_M \bar{\Delta}_h F(x) dV_h \geq 0.$$

By the divergence theorem, we know that

$$\int_B \frac{\partial F}{\partial \nu}(x) dA_h \geq 0.$$

Since $f = 0$ on B , then $\nu = \frac{\bar{\nabla} f}{|\bar{\nabla} f|}$ and $F(x) = \frac{1}{2}f_\nu^2 - \frac{1}{m}f$. From the fact that $\Delta_h f + f_{\nu\nu} + nH_h f_\nu = 1$ and $f = 0$ on B , we conclude that

$$0 \leq \int_B F_\nu dA_h = \int_B (f_\nu f_{\nu\nu} - \frac{1}{m} f_\nu) dA_h = \int_B (\frac{m-1}{m} f_\nu - nH_h f_\nu^2) dA_h.$$

Thus we have that

$$\int_B H_h f_\nu^2 dA_h \leq \frac{m-1}{mn} V_h. \tag{2.2}$$

Here we use the following equation.

$$\int_B f_\nu dA_h = \int_M \bar{\Delta}_h f dV_h = V_h. \tag{2.3}$$

Finally, from (2.2), (2.3) and Schwarz inequality it follows that

$$\begin{aligned} V_h^2 &= \left(\int_B f_\nu dA_h \right)^2 = \left(\int_B (H_h^{\frac{1}{2}} f_\nu) H_h^{-\frac{1}{2}} dA_h \right)^2 \\ &\leq \int_B H_h f_\nu^2 dA_h \int_B H_h^{-1} dA_h \leq \frac{m-1}{mn} V_h \int_B H_h^{-1} dA_h. \end{aligned}$$

Thus we have proved the inequality (1.4).

If M is isometric to an Euclidean ball and h is constant, then it is easy to conclude that the equality sign in (1.4) holds. Now we assume conversely that the equality sign in (1.4) holds. In this case all the equalities hold in (2.1). Thus $\frac{\bar{\Delta}f}{n+1} = -\frac{\bar{\nabla}h\bar{\nabla}f}{m-n-1} = \frac{\bar{\Delta}_h f}{m}$. As the proof of Theorem 3 in [6], we know that $m = n + 1$, h is constant. In fact, we have the following equation

$$\bar{\Delta}f = -\frac{n+1}{m-n-1} \bar{\nabla}h\bar{\nabla}f. \quad (2.4)$$

If $m > n + 1$, then multiplying (2.4) with f and integrating on M with respect to $e^{\frac{n+1}{m-n-1}h} dV$, we obtain that $\int_M |\bar{\nabla}f|^2 e^{\frac{n+1}{m-n-1}h} dV = 0$. So f is constant, which is a contradiction since $\bar{\Delta}_h f = 1$. Thus by Ros's result we see that M is isometric to an Euclidean ball.

Proof of Theorem 1.3. Let f be the smooth solution of the Neumann problem

$$\begin{cases} \bar{\Delta}_h f &= 1 & \text{in } M, \\ f_\nu &= \frac{V_h}{A_h} & \text{on } B. \end{cases}$$

Suppose $F = \frac{1}{2} |\bar{\nabla}f|^2 - \frac{1}{m} f$, from the proof of Theorem 1.1, we know that

$$\bar{\Delta}_h F \geq 0. \quad (2.5)$$

Then by the divergence theorem, we know that $\int_B \frac{\partial F}{\partial \nu}(x) dA_h \geq 0$. On the other hand,

$$\begin{aligned} F_\nu &= \sum_{i=1}^{n+1} f_{i\nu} f_i - \frac{1}{m} f_\nu \\ &= \sum_{i=1}^n f_{\nu i} f_i - \Pi_{ij} f_i f_j + f_{\nu\nu} f_\nu - \frac{1}{m} f_\nu \\ &\leq f_{\nu\nu} f_\nu - \frac{1}{m} f_\nu \\ &= (1 - \Delta_h f - n H_h f_\nu) f_\nu - \frac{1}{m} f_\nu \\ &= \frac{m-1}{m} f_\nu - \Delta_h f f_\nu - n H_h f_\nu^2. \end{aligned} \quad (2.6)$$

Thus from the boundary condition and the divergence theorem, we have that

$$\int_B H_h dA_h \leq \frac{m-1}{nm} \frac{A_h^2}{V_h}. \quad (2.7)$$

If M is isometric to an Euclidean ball and h is constant, then it is easy to conclude that the equality sign in (1.5) holds. Now we assume conversely that the equality sign in (1.5) holds. As

the proof of Theorem 1.1, we obtain that $\frac{\bar{\Delta}f}{n+1} = -\frac{\bar{\nabla}h\bar{\nabla}f}{m-n-1} = \frac{\bar{\Delta}_hf}{m}$. We claim that $m = n + 1$, h is constant. In fact, we have the following equation

$$\bar{\Delta}f = -\frac{n+1}{m-n-1}\bar{\nabla}h\bar{\nabla}f.$$

If $m > n + 1$, then integrating the above equation on M with respect to $e^{\frac{n+1}{m-n-1}h}dV$, we obtain that $\int_B f_\nu e^{\frac{n+1}{m-n-1}h}dA = 0$. This is a contradiction since $f_\nu = \frac{V_h}{A_h}$. On the other hand, if the equality sign in (1.5) holds, then the equalities in (2.5) and (2.6) hold. Which implies that $\bar{\nabla}^2 f = \frac{\bar{\Delta}f}{n+1}g$ and $\Pi(\bar{\nabla}f, \bar{\nabla}f) = 0$, i.e.

$$\begin{aligned}\bar{\nabla}^2 f &= \frac{1}{n+1}g \quad \text{in } M; \\ f &= \text{constant} \quad \text{on } B.\end{aligned}$$

Then by Lemma 3 in [13] we see that M is isometric to an Euclidean ball.

The proof of Theorem 1.4 is almost following Ros's method [12] and the first variation formulae of weighted volume and perimeter [15]. We include here a proof for the sake of completeness.

Proof of Theorem 1.4. Given a smooth function u on $\partial\Omega$, we consider the normal variation of $\partial\Omega$ defined by $\varphi_t : \partial\Omega \rightarrow M$, $\varphi_t(p) = \text{Exp}_p(tu(p)N(p))$, where Exp is the exponential map of M . φ_t determine a variation of Ω , Ω_t for $|t| < \epsilon$. Let $V_h(t) = V_h(\Omega_t)$ and $A_h(t) = A_h(\partial\Omega_t)$. The first variation formulae of the functionals above are given by

$$\begin{aligned}A'_h(0) &= n \int_{\partial\Omega} u H_h dA_h \\ V'_h(0) &= \int_{\partial\Omega} u dA_h.\end{aligned}$$

By hypothesis we show

$$\frac{d}{dt} \Big|_{t=0} \frac{A_h(t)^m}{V_h(t)^{m-1}} = 0,$$

or equivalently

$$\int_{\partial\Omega} u((mnV_h H_h - (m-1)A_h))dA_h = 0, \text{ for any } u.$$

Then $H_h = \frac{(m-1)A_h}{mnV_h}$. Therefore from Corollary 1.2 Ω is isometric to an Euclidean ball and h is constant.

3 Proof of Theorem 1.6, 1.7 and 1.11

Proof of Theorem 1.6. Let u be an eigenfunction corresponding to the first nonzero eigenvalue λ_1 of the drifting Laplacian of B :

$$\Delta_h u = -\lambda_1 u.$$

Let f be the smooth solution of the Dirichlet problem:

$$\begin{cases} \bar{\Delta}_h f &= 0 & \text{in } M, \\ f &= u & \text{on } B. \end{cases}$$

As the proof of Theorem 1.1, we show that

$$\begin{aligned} \frac{1}{2}\bar{\Delta}_h|\bar{\nabla}f|^2 &= |\bar{\nabla}^2f|^2 + \bar{\nabla}\bar{\Delta}_hf\bar{\nabla}f + Ric_h(\bar{\nabla}f, \bar{\nabla}f) \\ &\geq |\bar{\nabla}^2f|^2 + \frac{(\bar{\nabla}h \cdot \bar{\nabla}f)^2}{m-n-1} \geq 0. \end{aligned} \quad (3.1)$$

Then by the divergence theorem, we know that

$$\int_B \frac{\partial}{\partial \nu} \left(\frac{1}{2} |\bar{\nabla}f|^2 \right) dA_h \geq 0.$$

On the other hand,

$$\begin{aligned} \frac{\partial}{\partial \nu} \left(\frac{1}{2} |\bar{\nabla}f|^2 \right) &= \sum_{i=1}^n f_{\nu i} f_i - \Pi_{ij} f_i f_j + f_{\nu \nu} f_\nu \\ &= \sum_{i=1}^n f_{\nu i} u_i - \Pi_{ij} u_i u_j - \Delta_h u f_\nu - n H_h f_\nu^2 \\ &\leq \sum_{i=1}^n f_{\nu i} u_i - c |\nabla u|^2 + \lambda_1 u f_\nu - (m-1) c f_\nu^2. \end{aligned} \quad (3.2)$$

Thus we deduce that

$$\int_B \left(- \sum_{i=1}^n f_{\nu i} u_i + c |\nabla u|^2 - \lambda_1 u f_\nu + (m-1) c f_\nu^2 \right) dA_h \leq 0. \quad (3.3)$$

From the equation of the eigenfunction, we see that

$$\begin{aligned} 0 &\geq \int_B \left\{ - \sum_{i=1}^n f_{\nu i} u_i + c |\nabla u|^2 - \lambda_1 u f_\nu + (m-1) c f_\nu^2 \right\} dA_h \\ &= \int_B \left\{ \Delta_h u f_\nu + c \lambda_1 u^2 - \lambda_1 u f_\nu + (m-1) c f_\nu^2 \right\} dA_h \\ &= \int_B \left\{ c \lambda_1 u^2 - 2 \lambda_1 u f_\nu + (m-1) c f_\nu^2 \right\} dA_h \\ &= \int_B \left\{ (m-1) c \left(f_\nu - \frac{\lambda_1}{(m-1)c} u \right)^2 + \lambda_1 \left(c - \frac{\lambda_1}{(m-1)c} \right) u^2 \right\} dA_h \\ &\geq \lambda_1 \left(c - \frac{\lambda_1}{(m-1)c} \right) \int_B u^2 dA_h. \end{aligned}$$

Thus we have

$$\lambda_1 \geq (m-1)c^2.$$

If M is isometric to an Euclidean ball of radius $\frac{1}{c}$ and h is a constant, then $m = n + 1$, $\lambda_1(\Delta_h) = \lambda_1(\Delta) = nc^2 = (m-1)c^2$. Now we assume that $\lambda_1(\Delta_h) = (m-1)c^2$. In this case, the equality signs in (3.1), (3.2) and (3.3) hold. In particular, we see

$$\bar{\nabla}^2 f = 0, H_h = \frac{(m-1)}{n} c, f_\nu = \frac{\lambda_1}{(m-1)c} u = cu.$$

Since f is the first eigenfunction on B , then f is not a constant. Therefore from $\bar{\nabla}^2 f = 0$, we know $|\bar{\nabla} f|^2$ is a constant. By scaling, we assume $|\bar{\nabla} f|^2 = 1$. Thus we see that

$$A_h = \int_B |\bar{\nabla} f|^2 dA_h = \int_B (|\nabla u|^2 + f_\nu^2) dA_h = \int_B (\lambda_1 u^2 + f_\nu^2) dA_h = mc^2 \int_B u^2 dA_h. \quad (3.4)$$

On the other hand, since

$$\frac{1}{2} \bar{\Delta}_h(f^2) = |\bar{\nabla} f|^2 + f \bar{\Delta}_h f = 1,$$

then by the divergence theorem we know

$$V_h = \int_M \frac{1}{2} \bar{\Delta}_h(f^2) dV_h = \int_B u f_\nu dA_h = c \int_B u^2 dA_h. \quad (3.5)$$

Combining (3.4) and (3.5), we obtain

$$H_h = \frac{(m-1)}{n} c = \frac{(m-1)A_h}{mnV_h}.$$

So from Corollary 1.2 we know that M is isometric to an Euclidean ball and h is a constant. Since $\lambda_1(\Delta) = nc^2$, then the radius of M is $\frac{1}{c}$.

Proof of Theorem 1.7. Let u be an eigenfunction corresponding to the first nonzero eigenvalue λ_1 of the drifting Laplacian of B :

$$\Delta_h u = -\lambda_1 u.$$

Let f be the smooth solution of the Dirichlet problem:

$$\begin{cases} \bar{\Delta}_h f &= 0 & \text{in } M, \\ f &= u & \text{on } B. \end{cases}$$

As the proof of Theorem 1.6, we have that

$$\begin{aligned} \frac{1}{2} \bar{\Delta}_h |\bar{\nabla} f|^2 &= |\bar{\nabla}^2 f|^2 + \bar{\nabla} \bar{\Delta}_h f \bar{\nabla} f + Ric_h(\bar{\nabla} f, \bar{\nabla} f) \\ &\geq |\bar{\nabla}^2 f|^2 + \frac{(\bar{\nabla} h \cdot \bar{\nabla} f)^2}{m-n-1} \geq 0. \end{aligned} \quad (3.6)$$

Then by the divergence theorem, we know that

$$\int_B \frac{\partial}{\partial \nu} \left(\frac{1}{2} |\bar{\nabla} f|^2 \right) dA_h \geq 0.$$

On the other hand,

$$\begin{aligned} \frac{\partial}{\partial \nu} \left(\frac{1}{2} |\bar{\nabla} f|^2 \right) &= \sum_{i=1}^n f_{\nu i} u_i - \Pi_{ij} u_i u_j - \Delta_h u f_\nu - n H_h f_\nu^2 \\ &\leq \sum_{i=1}^n f_{\nu i} u_i - c |\nabla u|^2 + \lambda_1 u f_\nu - \frac{\lambda_1}{c} f_\nu^2. \end{aligned} \quad (3.7)$$

Thus we obtain that

$$0 \geq \int_B \left\{ - \sum_{i=1}^n f_{\nu i} u_i + c |\nabla u|^2 - \lambda_1 u f_\nu + \frac{\lambda_1}{c} f_\nu^2 \right\} dA_h \quad (3.8)$$

$$\begin{aligned}
&= \int_B \{ \Delta_h u f_\nu + c \lambda_1 u^2 - \lambda_1 u f_\nu + \frac{\lambda_1}{c} f_\nu^2 \} dA_h \\
&= \int_B \{ c \lambda_1 u^2 - 2 \lambda_1 u f_\nu + \frac{\lambda_1}{c} f_\nu^2 \} dA_h \\
&= \int_B \frac{\lambda_1}{c} (cu - f_\nu)^2 dA_h \geq 0.
\end{aligned}$$

Thus the equalities sign in (3.6), (3.7) and (3.8) hold. In particular, we see

$$\bar{\nabla}^2 f = 0, H_h = \frac{\lambda_1}{nc}, f_\nu = cu.$$

As the proof of Theorem 1.6, we assume $|\bar{\nabla} f|^2 = 1$. Thus we deduce that

$$A_h = \int_B |\bar{\nabla} f|^2 dA_h = \int_B (|\nabla u|^2 + f_\nu^2) dA_h = \int_B (\lambda_1 u^2 + f_\nu^2) dA_h = (\lambda_1 + c^2) \int_B u^2 dA_h. \quad (3.9)$$

On the other hand, since

$$\frac{1}{2} \bar{\Delta}_h (f^2) = |\bar{\nabla} f|^2 + f \bar{\Delta}_h f = 1,$$

then by the divergence theorem we know

$$V_h = \int_M \frac{1}{2} \bar{\Delta}_h (f^2) dV_h = \int_B u f_\nu dA_h = c \int_B u^2 dA_h. \quad (3.10)$$

Combining (3.9) and (3.10), we obtain

$$\frac{A_h}{V_h} = \frac{\lambda_1 + c^2}{c}. \quad (3.11)$$

Notice that $H_h \geq \frac{\lambda_1}{nc}$, then from Theorem 1.1, we see that

$$\lambda_1 \leq \frac{(m-1)c}{m} \cdot \frac{A_h}{V_h}. \quad (3.12)$$

By (3.11) and (3.12), we obtain that

$$\lambda_1 \leq (m-1)c^2.$$

Thus we have completed the proof of Theorem 1.7.

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